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Inequivalence of representations of commutation relations obtained by orthogonal transformations in field theory

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Abstract. It is shown that, except under very special conditions, orthogonal transformations on the field coordinates of a Bose–Einstein field cannot be represented as transformations in the Hilbert space of the boson field. The proof is restricted to rotations which are products of commuting rotations in two-dimensional subspaces.

1. Introduction

It is generally agreed upon that most of the mathematical difficulties of quantum field theory originate in the fact that fields have an infinite number of degrees of freedom. As a consequence of this, canonical transformations often lead to inequivalent representations of the commutation relations. Representations of commutation relations are said to be inequivalent if there exists no unitary transformation connecting the new coordinates and momenta to the old ones according to

$$q_j' = Uq_jU^{-1} \tag{1}$$

$$p_j' = Up_jU^{-1}. \tag{2}$$

These inequivalent representations were first noticed by Friedrichs (1953) and by van Hove (1952) and have been studied by Wightmann and Garding (1954) and by Wightmann and Schweber (1955). Representations encountered by van Kampen (1951) in diagonalizing the Hamiltonian of a nonrelativistic model of quantum electrodynamics can also be shown to be inequivalent. A comprehensive discussion of this general topic has been given by Haag (1960).

The purpose of this paper is to point out that, except under very strict conditions, even orthogonal transformations on the coordinates of a Bose–Einstein field lead to inequivalent representations. This result points out the inherent difficulty in the theory, since an orthogonal transformation is merely a rotation in the countably infinite dimensional space of the boson field coordinates.

2. Orthogonal transformations

A general orthogonal transformation may be written

$$q_i' = \sum_j \alpha_{ij} q_j \tag{3}$$

$$p_i' = \sum_j \alpha_{ij} p_j \tag{4}$$

where the α_{ij} satisfy the relation

$$\sum_k \alpha_{ik}\alpha_{jk} = \delta_{ij}. \tag{5}$$

In analogy to the three dimensional case, one might try to form the operator U appearing in (1) and (2) as follows:

$$U = \exp(-\mathrm{i} \sum_{k,l} \beta_{kl} q_k p_l) \tag{6}$$

with

$$\beta_{kl} = -\beta_{lk}. \tag{7}$$

Then (1) and (2) respectively, yield

$$q_j' = q_j + \sum_k \beta_{jk} q_k + \frac{1}{2!} \sum_{l,m} \beta_{jm} \beta_{ml} q_l + \dots \tag{8}$$

$$p_j' = p_j + \sum_k \beta_{jk} p_k + \frac{1}{2!} \sum_{l,m} \beta_{jm} \beta_{ml} p_l + \dots . \tag{9}$$

Comparison of (3) and (4) with (8) and (9) yields

$$\alpha_{ij} = (\mathrm{e}^{\beta})_{ij} \tag{10}$$

where β is an infinite dimensional square matrix having elements β_{ij}. Relations (7) and (10) guarantee the orthogonality of the transformation (3) and (4), since

$$\sum_k \alpha_{ik} \alpha_{jk} = \sum_k (\mathrm{e}^{\beta})_{ik} (\mathrm{e}^{\beta})_{jk} = \sum_k (\mathrm{e}^{\beta})_{ik} (\mathrm{e}^{-\beta})_{kj} = \delta_{ij}. \tag{11}$$

3. Restriction to the case of commuting two-dimensional rotations

In order to simplify the mathematics, we consider matrices U which are products of two dimensional rotations:

$$\begin{aligned} U &= \prod_{k<l} U_{kl} \\ U_{kl} &= \exp\{-\mathrm{i}\beta_{kl}(q_k p_l - q_l p_k)\} \\ \beta_{kl} &= \lambda_n \qquad k = l-1 = 2n-1 \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{12}$$

Given that the Bose–Einstein system under consideration has the Hamiltonian

$$H = \sum_i \tfrac{1}{2}(p_i^2 + w_i^2 q_i^2) \tag{13}$$

the ground state for the first two oscillators is given by

$$\psi_0 = \left(\frac{\omega_1}{\pi}\right)^{1/4} \left(\frac{\omega_2}{\pi}\right)^{1/4} \exp\left(-\frac{\omega_1}{2} q_1^2\right) \exp\left(-\frac{\omega_2}{2} q_2^2\right). \tag{14}$$

Since U_{12} is the operator that rotates q_1 into q_1' and q_2 into q_2', we have

$$\begin{aligned} U_{12}\psi_0 &= \left(\frac{\omega_1}{\pi}\right)^{1/4} \left(\frac{\omega_2}{\pi}\right)^{1/4} \exp\left(-\frac{\omega_1}{2} q_1'^2\right) \exp\left(-\frac{\omega_2}{2} q_2'^2\right) \\ &= \left(\frac{\omega_1}{\pi}\right)^{1/4} \left(\frac{\omega_2}{\pi}\right)^{1/4} \exp\left\{-\frac{\omega_1}{2} (\alpha_{11} q_1 + \alpha_{12} q_2)^2\right\} \exp\left\{-\frac{\omega_2}{2} (\alpha_{21} q_1 + \alpha_{22} q_2)^2\right\}. \end{aligned} \tag{15}$$

Straightforward computation shows that

$$\langle\psi_0|U_{12}|\psi_0\rangle = \frac{2(\omega_1\omega_2)^{1/2}}{\{2\omega_1\omega_2+\omega_1\omega_2(\alpha_{11}{}^2+\alpha_{22}{}^2)+\omega_1{}^2\alpha_{12}{}^2+\omega_2{}^2\alpha_{21}{}^2\}^{1/2}}. \tag{16}$$

Thus, $\langle\psi_0|U_{12}|\psi_0\rangle = 1$ if $\omega_1 = \omega_2$ or if $\alpha_{12} = \alpha_{21} = 0$. We now consider transformations such that $\omega_l - \omega_k = \epsilon_n$; $k = l-1 = 2n-1$ with

$$\epsilon_n \ll \omega_l \simeq \omega_k. \tag{17}$$

Thus

$$\langle\psi_0|U_{12}|\psi_0\rangle = 1 - \frac{\alpha_{21}{}^2}{8\omega_1{}^2}\epsilon_1{}^2 + O(\epsilon_1)^4. \tag{18}$$

The transformation U therefore yields:

$$\langle\psi_0|U|\psi_0\rangle = \prod_n \left(1 - \frac{\alpha^2_{2n,2n-1}}{8\omega^2_{2n-1}}\epsilon_n{}^2\right). \tag{19}$$

A necessary condition for this to converge is

$$\sum_n \frac{\alpha^2_{2n,2n-1}}{8\omega^2_{2n-1}}\epsilon_n{}^2 < \infty. \tag{20}$$

For quantization of the boson field in a periodicity volume L^3, $\alpha_{ij} \propto 1/L^{3/2}$ because of (11). Hence if $L^{3/2}|\alpha_{2n,2n-1}| > \delta > 0$ for all n, and if the density of states for the boson field is $L^3/8\pi^3$, then $\langle\psi_0|U|\psi_0\rangle = 0$ unless

$$\int_0^\infty \epsilon^2(\omega)\,d\omega < \infty.$$

Hence $\langle\psi_0|U|\psi_0\rangle = 0$ unless $\epsilon_n \to 0$ faster than $1/n^{1/2}$. Hence, if $|\psi_i\rangle$ and $|\psi_j\rangle$ are states of finitely many quanta, it follows immediately that $\langle\psi_j|U|\psi_i\rangle = 0$ for all $|\psi_i\rangle$ and $|\psi_j\rangle$. Hence unless $\epsilon_n \to 0$ faster than $1/n^{1/2}$, U is not an operator in the Hilbert space of the Bose–Einstein field.

Similar theorems have been proved by Segal (1958). The significance of the present result, however, is that even mere rotations in the space of the field coordinates can seldom be represented as transformations in the original Hilbert space. Such rotations can, in practice, be quite useful; an example is van Kampen's treatment of light scattering (1951).

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